# Polyharmonicity and algebraic support of measures

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**Abstract**

Our main result states that two measures $\mu$ and $\nu$ with bounded support contained in the zero set of a polynomial $P(x)$ are equal if they coincide on the subspace of all polynomials of polyharmonic degree $N_P$ where the natural number $N_P$ is explictly computed by the properties of the polynomial $P(x)$. The method of proof depends on a definition of a multivariate Markov transform which another major objective of the present paper. The classical notion of orthogonal polynomial of second kind is generalized to the multivariate setting: it is a polyharmonic function which has similar features as in the one-dimensional case.

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## 1 Introduction

Recall that a complex-valued function $f$ defined on a domain $G$ in the euclidean space $\mathbb{R}^n$ is *polyharmonic of order* $N$ if $f$ is $2N$-times continuously differentiable and

$$\Delta^N f(x) = 0 \text{ for all } x \in G$$

where $\Delta^N$ is the $N$-th iterate of the Laplace operator $\Delta = \frac{\partial^2}{\partial x_1^2} + ... + \frac{\partial^2}{\partial x_n^2}$. For $N = 1$ this class of functions are just the harmonic functions, while for $N = 2$ the term biharmonic function is used which is important in elasticity theory. Fundamental work about polyharmonic functions is due to E. Almansi [2], M. Nicolesco (see e.g. [27]) and N. Aronszajn [3], and still this is an area of active research, see e.g. [7], [8],[9], [13], [18],[20], [25], [30], [31]. Polyharmonic functions are also important in applied mathematics, e.g. in approximation theory, radial basis functions and wavelet analysis, see e.g. [5], [21], [22], [23], [26].

In this paper we address the following question: suppose that $P(x)$ is a polynomial, and that $\mu$ and $\nu$ are signed measures which have support in the zero set $K_P$ of the polynomial $P$, i.e. in the set

$$K_P(R) := \{x \in \mathbb{R}^n : P(x) = 0 \text{ and } |x| \leq R\}.$$

Under which conditions do $\mu$ and $\nu$ coincide? As motivating example consider the polynomial $P(x) = |x|^2 - 1$ where $|x| := r(x) := \sqrt{x_1^2 + ... + x_n^2}$ is the euclidean norm in $\mathbb{R}^n$. It is well known that two measures $\mu$ and $\nu$ with support in the unit sphere $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ coincide if they are equal on the set of all *harmonic* polynomials. We shall show that two measures $\mu$ and $\nu$ with support in $K_P(R)$ are equal if the moments $\mu(f)$ an $\nu(f)$ are equal for polyharmonic polynomials $f$ of a certain degree $N_P$ which depends on the polynomial $P$. In order to formulate this precisely, let us introduce the *polyharmonic degree* $d(f)$ defined by

$$d(f) := \min\left\{N \in \mathbb{N}_0 : \Delta^{N+1}(f) = 0\right\} \tag{1}$$

Note that$f$ has **polyharmonic degree** $\leq N$ if and only if $f$ is of **polyharmonic order** $N+1$.

Let us denote by $\mathcal{P}$ set of all polynomials. One of the main results of this paper reads as follows:

**Theorem 1** *Let $P(x)$ be a polynomial and define*

$$N_P := \sup\{d(P \cdot h) : h \in \mathcal{P} \text{ is a harmonic polynomial}\}.$$

*Let $\mu$ and $\nu$ be measures with support contained in the set $K_P(R)$ for some $R > 0$. Then $\mu \equiv \nu$ if and only if $\int h d\mu = \int h d\nu$ for all polynomials $h$ in the subspace*

$$U_{N_P} := \left\{Q \in \mathcal{P} : \ \Delta^{N_P} Q = 0\right\}.$$

It is easy to see that $N_P$ is lower or equal to the total degree of the polynomial $P(x)$. In the appendix we shall give a procedure to determine the number $N_P$ explicitly.

An application of the Hahn-Banach theorem shows us the following consequence of Theorem 1: the space $U_{N_P}$ is dense in the space $C(K_P(R), \mathbb{C})$ of all continuous complex-valued functions on the compact space $K_P(R)$ endowed with the supremum norm, see Corollary 17. We call the reader's attention to this interesting result which may be compared with the density results for solutions to $\Delta^p h = 0$ in $C(K)$ for compacts $K$, obtained with the techniques of Potential theory in the 1970s; see [14], [15] and the references therein.

It is also instructive to consider the statement of Theorem 1 for the univariate case $n = 1$, so $P$ is a polynomial of degree $N$, and $P^{-1}(0)$ has at most $N$ elements. Note that $\Delta^N Q = \frac{d^{2N}}{dx^{2N}} Q = 0$ if and only if $Q$ is a polynomial of degree $\leq 2N - 1$. Hence, Theorem 1 says that two non-negative measures $\mu$ and $\nu$ with support in $P^{-1}(0)$ are equal if and only if

$$\int x^s d\mu = \int x^s d\nu \text{ for all } s \leq 2N - 1.$$

So Theorem 1 can be seen as a generalization of a simple univariate statement based upon the Polyharmonic paradigm as presented in [21, chapter 1.5].

The proof of Theorem 1 will be a by-product of our investigation of the so-called *multivariate Markov transform* which we will introduce below and which we consider as a suitable generalization of the univariate *Markov transform*, an important tool in the classical moment problem and its applications to Spectral theory. Recall that the *Markov transform*[1] of a finite measure $\sigma$ with support in the interval $[-R, R]$ is defined on the upper half–plane by the formula

$$\widehat{\sigma}(\zeta) := \int_{-R}^{R} \frac{1}{\zeta - x} d\sigma(x) \text{ for } \operatorname{Im}\zeta > 0, \tag{2}$$

see e.g. [1, Chapter 2], [28, Chapter 2.6]. Let us recall a central result called Markov's theorem: the $N-$th Padé approximant $\pi_N(\zeta) = Q_N(\zeta)/P_N(\zeta)$ of the asymptotic expansion of $\widehat{\sigma}(\zeta)$ at infinity converges compactly in the upper half plane to $\widehat{\sigma}(\zeta)$; here the polynomial $P_N$ is the $N$-th orthogonal polynomial with respect to the measure $\sigma$ and $Q_N$ is the *orthogonal polynomial of the second kind* with respect to the measure $\sigma$ given through the formula

$$Q_N(\zeta) = \int_{-\infty}^{\infty} \frac{P_N(\zeta) - P_N(x)}{\zeta - x} d\sigma(x). \tag{3}$$

Further, to each $\pi_N(\zeta)$ there corresponds a (non–negative) measure $\sigma_N$ with support in the zeros of the nominator $P_N$, thus leading to a proof of the famous Gauß quadrature formula.

Our definition of a multivariate Markov transform depends on the work of *N. Aronszajn* [3] on polyharmonic functions, and of *L.K. Hua* [16] about harmonic analysis on Lie groups; the definition is related to the Poisson formula for the ball $B_R := \{x \in \mathbb{R}^n : |x| < R\}$ which we recall now: Let $R > 0$ and $h$ be a function harmonic in the ball $B_R$ and continuous on the closure $\overline{B_R}$; then for any $x \in \mathbb{R}^n$ with $|x| < R$

$$h(x) = \frac{1}{\omega_n} \int_{\mathbb{S}^{n-1}} \frac{\left(R^2 - |x|^2\right) R^{n-2}}{r(R\theta - x)^n} h(R\theta) d\theta, \tag{4}$$

where $\omega_n$ denotes the area of $\mathbb{S}^{n-1}$, $\theta \in \mathbb{S}^{n-1}$, $y = R\theta$, and $r(x)$ is the euclidean norm of $x$. Note that for fixed $x$ with $|x| < R$ the function $\rho \longmapsto r(\rho\theta - x)$ defined for $\rho \in \mathbb{R}$ with $|\rho| > R$ has an analytic continuation for $\zeta \in \mathbb{C}$ with $|\zeta| > R$, so we can write $r(\zeta\theta - x)$ for $\zeta \in \mathbb{C}$ with $|\zeta| > R$. The following *Cauchy type integral formula*, proved in [3, p. 125], is important for our approach: for any polynomial $u(x)$ and for any $|x| < R$ the following identity holds

$$u(x) = \frac{1}{2\pi i \omega_n} \int_{\Gamma_R} \int_{\mathbb{S}^{n-1}} \frac{\zeta^{n-1}}{r(\zeta\theta - x)^n} u(\zeta\theta) d\theta d\zeta \tag{5}$$

[1] In some recent works in Approximation theory, Potential theory, and Probability theory this function is called the *Markov function* of a measure, see e.g. [32] or [12]. On the other hand apparently Widder [35] was the first who has given the name *Stieltjes transform* to this function. If $\mu$ has infinite support the transform is also called Stieltjes transform. This tradition has been followed by Akhiezer [1] and other Russian mathematicians.

where the contour $\Gamma_R(t) = R \cdot e^{it}$ for $t \in [0, 2\pi]$. A similar result is also valid for holomorphic functions $u$ defined on the so-called harmonicity hull of $B_R$; since we need (5) only for polynomials we refer the reader to [3, p. 125] for details.

Assume now that $\mu$ is a measure with support in the closed ball $\{x \in \mathbb{R}^n : |x| \leq R\}$. The *multivariate Markov transform* $\widehat{\mu}$ of $\mu$ is a function defined for all $\theta \in \mathbb{S}^{n-1}$ and all $\zeta \in \mathbb{C}$ with $|\zeta| > R$ by the formula

$$\widehat{\mu}(\zeta, \theta) = \frac{1}{\omega_n} \int_{\mathbb{R}^n} \frac{\zeta^{n-1}}{r(\zeta\theta - x)^n} d\mu(x). \tag{6}$$

Since $\zeta \mapsto r(\zeta\theta - x)$ has no zeros for $|\zeta| > R$ the function $\zeta \mapsto \widehat{\mu}(\zeta, \theta)$ is defined for all $|\zeta| > R$. In the first Section we shall show that the multivariate Markov transform $\widehat{\mu}$ determines the measure $\mu$ uniquely, cf. Theorem 3.

Our second main innovation is the introduction of the notion of the *function* $Q_P(\zeta, \theta)$ *of the second kind* with respect to a given polynomial $P(x)$ which is the multivariate analogue of (3), defined by

$$Q_P(\zeta, \theta) = \int_{\mathbb{R}^n} \frac{P(\zeta\theta) - P(x)}{r(\zeta\theta - x)^n} \zeta^{n-1} d\mu(x) \tag{7}$$

for all $|\zeta| > R, \theta \in \mathbb{S}^{n-1}$. Let us emphasize that $Q_P$ is in general *not* a polynomial. However, we shall show the surprising and interesting result that the function $r\theta \mapsto r^{-(n-1)} Q_P(r\theta)$ is a *polyharmonic* function of order $\leq \deg P(x)$ where deg denotes the usual total degree of a polynomial.

One further *main result* of the paper, Theorem 13, is concerned with measures $\mu$ having their supports in algebraic sets: Let us assume that the measure $\mu$ has support in $K_P(R)$. Then the Markov transform $\widehat{\mu}$ has the representation

$$\widehat{\mu}(\zeta, \theta) = \frac{Q_P(\zeta, \theta)}{P(\zeta\theta)} \qquad \text{for } |\zeta| > R, \tag{8}$$

where $Q_P$ is the function of second kind with respect to $P(x)$. The reverse statement holds as well, i.e. if the measure $\mu$ with $\operatorname{supp}(\mu) \subset \overline{B_R}$ satisfies (8) for some polynomial $P$ where $Q_P$ is defined by (7), then $\operatorname{supp}(\mu) \subset K_P(R)$. By means of these characterizations we can deduce our main result Theorem 1.

# 2 The multivariate Markov transform

Recall that the univariate Markov transform has, for $|\zeta| > R$, the asymptotic expansion

$$\widehat{\sigma}(\zeta) = \sum_{k=0}^{\infty} \frac{1}{\zeta^{k+1}} \int_{-\infty}^{\infty} t^k d\sigma(t). \tag{9}$$

Let $\Gamma_R$ denote the contour in $\mathbb{C}$ defined by $\Gamma_R(t) = R \cdot e^{it}$ for $t \in [0, 2\pi]$. By means of standard facts from complex analysis the following identity may be proved,

$$M(p) := \frac{1}{2\pi i} \int_{\Gamma_{R_1}} p(\zeta)\, \widehat{\sigma}(\zeta)\, d\zeta = \int_{-R}^{R} p(x)\, d\sigma(x) \tag{10}$$

for all polynomials $p$ and any $R_1 > R$.

In this section we want to show that similar results hold for the multivariate Markov transform $\widehat{\mu}$; in particular the following is the analogue of formula (10) in the multivariate case:

**Proposition 2** *Let $\mu$ be a signed measure over $\mathbb{R}^n$ with support in $\overline{B_R}$ and let $R_1 > R$. Then for every polnomial $P(x)$*

$$M_\mu (P) := \frac{1}{2\pi i} \int_{\Gamma_{R_1}} \int_{\mathbb{S}^{n-1}} P(\zeta\theta) \widehat{\mu}(\zeta, \theta) \, d\zeta d\theta = \int_{\mathbb{R}^n} P(x) \, d\mu(x) . \tag{11}$$

**Proof.** Replace $\widehat{\mu}(\zeta, \theta)$ in (11) by (6) and interchange integration. Then

$$M_\mu (P) = \int_{\mathbb{R}^n} \frac{1}{2\pi i \omega_n} \int_{\Gamma_{R_1}} \int_{\mathbb{S}^{n-1}} P(\zeta\theta) \frac{\zeta^{n-1}}{r(\zeta\theta - x)^n} d\zeta d\theta d\mu(x) . \tag{12}$$

According to (5) we obtain $M_\mu (P) = \int P(x) \, d\mu(x)$ . ■

**Theorem 3** *Let $\mu, \nu$ be finite signed measures over $\mathbb{R}^n$ with compact support. If the multivariate Markov transforms of $\mu$ and $\nu$ coincide for large $\zeta$, i.e., if there exists $R > 0$ such that $\widehat{\mu}(\zeta, \theta) = \widehat{\nu}(\zeta, \theta)$ for all $|\zeta| > R$ and for all $\theta \in \mathbb{S}^{n-1}$, then $\mu$ and $\nu$ are identical.*

**Proof.** Since the multivariate Markov transforms coincide for large $|\zeta|$ it is clear that the functionals $M_\mu$ and $M_\nu$ in (11) are identical by taking the radius $R_1$ of the path $\Gamma_{R_1}$ large enough. Then Proposition 2 shows that $\int P(x) \, d\mu(x) = \int P(x) \, d\nu(x)$ for all polynomials $P(x)$. Further we apply a standard argument: since $\mu$ and $\nu$ have compact supports we may apply the Stone–Weierstrass theorem according to which the polynomials are dense in the space $C(\operatorname{supp}(\mu) \cup \operatorname{supp}(\nu))$ which implies by the Hahn–Banach theorem that $\mu = \nu$. ■

Next we want to determine the asymptotic expansion of the multivariate Markov transform and we need some notations from harmonic analysis; for a detailed account we refer to [4] or [33]. Recall that a function $Y : \mathbb{S}^{n-1} \to \mathbb{C}$ is called a *spherical harmonic* of degree $k \in \mathbb{N}_0$ if there exists a *homogeneous harmonic* polynomial $P(x)$ of degree $k$ (in general, with complex coefficients) such that $P(\theta) = Y(\theta)$ for all $\theta \in \mathbb{S}^{n-1}$.[2] Throughout the paper we assume that $Y_{k,m}(x)$, $m = 1, ..., a_k$, is a basis of the set of all harmonic homogeneous polynomials of degree $k$ which are orthonormal with respect to scalar product

$$\langle f, g \rangle_{\mathbb{S}^{n-1}} := \int_{\mathbb{S}^{n-1}} f_m(\theta) \overline{g(\theta)} d\theta.$$

For a continuous function $f : \mathbb{S}^{n-1} \to \mathbb{C}$ we define the *Laplace-Fourier series* by

$$f(\theta) = \sum_{k=0}^{\infty} \sum_{m=1}^{a_k} f_{k,m} Y_{k,m}(\theta)$$

[2]One may restrict the attention to real valued spherical harmonics and this does not change the results essentially.

and $f_{k,m} = \int_{\mathbb{S}^{n-1}} f(\theta) \overline{Y_{k,m}(\theta)} d\theta$ are the *Laplace-Fourier coefficients* of $f$.

Using the *Gauss decomposition* of a polynomial (see Theorem 5.5 in [4]) it is easy to see that the system

$$|x|^{2t} Y_{k,m}(x), t, k \in \mathbb{N}_0, m = 1, ..., a_k$$

is a basis of the set of all polynomials. The numbers

$$c_{t,k,m} := \int_{\mathbb{R}^n} |x|^{2t} \overline{Y_{k,m}(x)} d\mu(x), \quad t, k \in \mathbb{N}_0, m = 1, ..., a_k \tag{13}$$

are sometimes called the *distributed moments*, see [17]. For a treatment and formulation of the *multivariate moment problem* we refer to [10], see also [34].

**Theorem 4** *Let $\mu$ be a signed measure over $\mathbb{R}^n$ with support in the closed ball $\overline{B_R}$. Then for all $|\zeta| > R$ and for all $\theta \in \mathbb{S}^{n-1}$ the following relation holds*

$$\widehat{\mu}(\zeta, \theta) = \sum_{t=0}^{\infty} \sum_{k=0}^{\infty} \sum_{m=1}^{a_k} \frac{Y_{k,m}(\theta)}{\zeta^{2t+k+1}} \int_{\mathbb{R}^n} |x|^{2t} \overline{Y_{k,m}(x)} d\mu(x) \tag{14}$$

**Proof.** A zonal harmonic of degree $k$ with pole $\theta \in \mathbb{S}^{n-1}$ is the unique spherical harmonic $Z_\theta^{(k)}$ of degree $k$ such that for all spherical harmonics $Y$ of degree $k$ the relation $Y(\theta) = \int_{\mathbb{S}^{n-1}} Z_\theta^{(k)}(\eta) Y(\eta) d\eta$ holds. Let $p_n(\theta, x) = \frac{1}{\omega_n} \frac{1-|x|^2}{|x-\theta|^n}$ be the Poisson kernel for $0 \leq |x| < 1 = |\theta|$. Theorem 2.10 in [33, p. 145] gives $p_n(\theta, x) = \sum_{k=0}^{\infty} |x|^k Z_\theta^{(k)}(x')$ for all $\theta, x' \in \mathbb{S}^{n-1}$, where $x = |x| \cdot x'$, $|x| < 1$. Lemma 2.8 in [33] shows that $Z_\theta^{(k)}(x') = \sum_{m=1}^{a_k} \overline{Y_{k,m}(x')} Y_{k,m}(\theta)$ where $x', \theta \in \mathbb{S}^{n-1}$, so

$$p_n(\theta, x) = \sum_{k=0}^{\infty} \sum_{m=1}^{a_k} |x|^k \overline{Y_{k,m}(x')} Y_{k,m}(\theta). \tag{15}$$

for $|x| < 1$. Let $R$ be as in the theorem, and replace now $x$ in (15) by $x/\rho$, $\rho \in \mathbb{R}$ such that $|x| < R < \rho$; one obtains that

$$\frac{1}{\omega_n} \frac{\rho^{n-2} \left(\rho^2 - |x|^2\right)}{r(\rho\theta - x)^n} = \sum_{k=0}^{\infty} \sum_{m=1}^{a_k} \frac{1}{\rho^k} \overline{Y_{k,m}(x)} Y_{k,m}(\theta). \tag{16}$$

The real variable $\rho$ can now be replaced by a complex variable $\zeta$ with $|\zeta| > R$. We multiply by $\zeta \left(\zeta^2 - |x|^2\right)^{-1}$, and integrate integrate over the closed ball $\overline{B_R}$ with respect to $\mu$. This gives

$$\widehat{\mu}(\zeta, \theta) = \sum_{k=0}^{\infty} \sum_{m=1}^{a_k} Y_{k,m}(\theta) \zeta^{-k+1} \int_{\mathbb{R}^n} \frac{\overline{Y_{k,m}(x)}}{\zeta^2 - |x|^2} d\mu(x), \tag{17}$$

and we have determined the Laplace-Fourier series of $\theta \longmapsto \widehat{\mu}(\zeta,\theta)$. Since $|\zeta| > R \geq |x|$ we can expand $1/(1-\frac{|x|^2}{\zeta^2})$ in a geometric series and we obtain

$$\widehat{\mu}(\zeta,\theta) = \sum_{k=0}^{\infty}\sum_{m=1}^{a_k} \frac{Y_{k,m}(\theta)}{\zeta^{k+1}} \int_{\mathbb{R}^n} \overline{Y_{k,m}(x)} \left( \sum_{t=0}^{\infty} \frac{|x|^{2t}}{\zeta^{2t}} \right) d\mu(x). \tag{18}$$

After interchanging summation and integration the claim is obvious. ■

# 3 The function of the second kind

In the following we want to a give a multivariate analogue of the polynomial of second kind. It turns out that in the multivariate case the corresponding definition does not lead to a polynomial but to a polyharmonic function $Q_P(\zeta,\theta)$ which is defined only for all $|\zeta| > R, \theta \in \mathbb{S}^{n-1}$.

**Definition 5** *Let $P(x)$ be a polynomial and $\mu$ be a non-negative measure with support in $\overline{B_R}$. Then the* function $Q_P(\zeta,\theta)$ of the second kind *is defined by*

$$Q_P(\zeta,\theta) = \frac{1}{\omega_n} \int_{\mathbb{R}^n} \frac{P(\zeta\theta) - P(x)}{r(\zeta\theta - x)^n} \zeta^{n-1} d\mu(x)$$

*for all $|\zeta| > R, \theta \in \mathbb{S}^{n-1}$. Similarly we define the function $R_P(\zeta,\theta)$ by*

$$R_P(\zeta,\theta) = \frac{1}{\omega_n} \int_{\mathbb{R}^n} \frac{P(x)}{r(\zeta\theta - x)^n} \zeta^{n-1} d\mu(x)$$

*for all $|\zeta| > R, \theta \in \mathbb{S}^{n-1}$.*

The last definitions immediately give the identity

$$P(\zeta\theta)\,\widehat{\mu}(\zeta,\theta) = Q_P(\zeta,\theta) + R_P(\zeta,\theta). \tag{19}$$

**Theorem 6** *Let $P(x)$ be a polynomial, $\mu$ be a signed measure with support in $\overline{B_R}$ and $Q_P(\zeta,\theta)$ the function of the second kind. Then for any $R_1 > R$ and for each polynomial $h(x)$*

$$\frac{1}{2\pi i} \int_{\Gamma_{R_1}} \int_{\mathbb{S}^{n-1}} h(\zeta\theta)\, Q_P(\zeta,\theta)\, d\zeta d\theta = 0. \tag{20}$$

**Proof.** Let us denote the integral in (20) by $I(h)$. By (19) we obtain that $I(h) = I_1(h) - I_2(h)$ where

$$I_1(h) = \frac{1}{2\pi i} \int_{\Gamma_{R_1}} \int_{\mathbb{S}^{n-1}} h(\zeta\theta)\, P(\zeta\theta)\, \widehat{\mu}(\zeta,\theta)\, d\zeta d\theta, \tag{21}$$

$$I_2(h) = \frac{1}{2\pi i \omega_n} \int_{\Gamma_{R_1}} \int_{\mathbb{S}^{n-1}} h(\zeta\theta) \int_{\mathbb{R}^n} \frac{P(x)}{r(\zeta\theta - x)^n} \zeta^{n-1} d\mu(x)\, d\zeta d\theta. \tag{22}$$

Proposition 2 yields $I_1(h) = \int_{\mathbb{R}^n} h(x) P(x) d\mu(x)$. Change the integration order in (22) and use formula (5). Then we obtain $I_2(h) = I_1(h)$, therefore $I(h) = 0$ which was our claim. ■

A similar argument as in the proof of formula (14) proves the following:

**Theorem 7** *The rest function $R_P(\zeta, \theta)$ has the asymptotic expansion*

$$\sum_{t=0}^{\infty} \sum_{k=0}^{\infty} \sum_{m=1}^{a_k} \frac{Y_{k,m}(\theta)}{\zeta^{2t+k+1}} \int_{\mathbb{R}^n} P(x) |x|^{2t} \overline{Y_{k,m}(x)} d\mu(x). \tag{23}$$

Let us consider now the Laurent series of the function $\zeta \mapsto R_P(\zeta, \theta)$: for $|\zeta| > R, \theta \in \mathbb{S}^{n-1}$ we can write

$$R_P(\zeta, \theta) = \sum_{s=0}^{\infty} r_s[P](\theta) \frac{1}{\zeta^{s+1}}. \tag{24}$$

From (23), by putting $s = 2t + k$, it follows that

$$r_s[P](\theta) = \sum_{t=0}^{[s/2]} \sum_{m=1}^{a_{s-2t}} Y_{s-2t,m}(\theta) \int_{\mathbb{R}^n} P(x) |x|^{2t} \overline{Y_{s-2t,m}(x)} d\mu(x). \tag{25}$$

Hence the coefficient function $r_s(P)$ is a sum of spherical harmonics with degree $\leq s$.

We can now formulate a characterization of orthogonality in asymptotic analysis:

**Theorem 8** *Let $\mu$ be a signed measure with compact support and $P(x)$ be a polynomial. Then $P$ is orthogonal to all polynomials of degree $< M$ with respect to $\mu$ if and only if*

$$r_0[P] = \ldots = r_{M-1}[P] = 0$$

*where $r_s[P]$ are the functions defined in (24)–(25).*

**Proof.** From (25) we see that $r_0(P) = \ldots = r_{M-1}(P) = 0$ if and only for all $s = 0, \ldots, M-1$

$$\int_{\mathbb{R}^n} P(x) |x|^{2t} \overline{Y_{s-2t,m}(x)} d\mu(x) = 0.$$

But the polynomials $|x|^{2t} Y_{s-2t,m}(x)$ with $s = 0, \ldots, M-1$, $t = 0, \ldots, [s/2], m = 1, \ldots, a_{s-2t}$, span up the space of polynomials of degree $\leq M-1$. ■

The next theorem, interesting in its own right, is not needed later, and therefore the proof will be omitted.

**Theorem 9** *Let $\mu$ be a signed measure with compact support and let $P(x)$ be a polynomial of degree $2N$. If $P$ is orthogonal to all polynomials of degree $\leq 2N$ and polyharmonic degree $< N$ then $r_0(P) = \ldots = r_{2N-1}(P) = 0$ and $r_{2N}(\theta)$ is constant.*

# 4 Polyharmonicity of the function of second kind

In this Section we want to show that the function $Q_P(\zeta,\theta)$ of the second kind, multiplied by $\zeta^{-(n-1)}$, is a polyharmonic function.

Recall that we have defined $N_P = \sup\{d(P\cdot h) : h \text{ harmonic polynomial}\}$ for a polynomial $P(x)$. In the Appendix we will show that $N_P \leq \deg P(x)$ and an explicit determination of $N_P$ will be given there as well.

**Proposition 10** *Let $Y_{k,m}, m = 1,...,a_k$, be an orthonormal basis of the space of all homogeneous harmonic polynomials. Then*

$$N_P := \sup_{k\in\mathbb{N}_0, m=1,\ldots,a_k} d(P(x) Y_{k,m}(x)). \tag{26}$$

**Proof.** Let us denote the right hand side by $M_P$. Then the inequality $M_P \leq N_P$ is trivial. For the converse let $h(x)$ be a harmonic polynomial and write $h(x) = \sum_{k=0}^{N}\sum_{m=1}^{a_k}\lambda_{k,m}Y_{k,m}(x)$. Then

$$d(P\cdot h) \leq \sup_{k\in\mathbb{N}_0, m=1,\ldots,a_k} d(P(x) Y_{k,m}(x)) \leq M_P.$$

■

Note that $N_P = \sup_{k\in\mathbb{N}_0, m=1,\ldots,a_k} d\left(P(x)\overline{Y_{k,m}(x)}\right)$ since $\overline{Y_{k,m}}$, $m = 1,...,a_k$ is an orthonormal basis as well. Now we determine the asymptotic expansion of the function of the second kind:

**Theorem 11** *Let $P(x)$ be a polynomial and $\mu$ be a signed measure with support in $\overline{B_R}$. Then $\theta \mapsto Q_P(\zeta,\theta)$, the function of the second kind, possesses a Laplace-Fourier series of the form*

$$Q_P(\zeta,\theta) = \sum_{k=0}^{\infty}\sum_{m=1}^{a_k}\frac{1}{\zeta^{k-1}}p_{k,m}\left(\zeta^2\right)Y_{k,m}(\theta) \tag{27}$$

*where $p_{k,m}(t)$ are univariate polynomials of degree strictly smaller than $N_{k,m} := d(P(x)Y_{k,m}(x))$. The function $Q_P(\zeta,\theta)$ of the second kind depends on those distributed moments*

$$\int_{\mathbb{R}^n} h(x)|x|^{2t}\,d\mu(x) \tag{28}$$

*where $t \leq \sup_{k\in\mathbb{N}_0}\deg p_{k,m}$ and $h(x)$ is a harmonic polynomial.*

**Proof.** For each fixed $\zeta$ with $|\zeta| > R$ the function $\theta \mapsto Q_P(\zeta,\theta)$ possesses a Laplace-Fourier expansion, say

$$Q_P(\zeta,\theta) = \sum_{k=0}^{\infty}\sum_{m=1}^{a_k} e_{km}(\zeta)\,Y_{k,m}(\theta)$$

Recall that $Q_P(\zeta,\theta) = P(\zeta\theta)\,\widehat{\mu}(\zeta,\theta) - R_P(\zeta,\theta)$. Formula (23) yields the Laplace-Fourier expansion of $\theta \mapsto R_P(\zeta,\theta)$ : in (23) one computes the sum over the

variable $t$ obtaining

$$R_P(\zeta,\theta)=\sum_{k=0}^{\infty}\sum_{m=1}^{a_k}Y_{k,m}(\theta)\frac{1}{\zeta^{k-1}}\int_{\mathbb{R}^n}\frac{P(x)\overline{Y_{k,m}(x)}}{\zeta^2-|x|^2}d\mu(x). \tag{29}$$

The Laplace-Fourier coefficients of $\theta\mapsto P(\zeta\theta)\widehat{\mu}(\zeta,\theta)$ are given through

$$f_{k,m}(\zeta):=\int_{\mathbb{S}^{n-1}}P(\zeta\theta)\widehat{\mu}(\zeta,\theta)\overline{Y_{k,m}(\theta)}d\theta. \tag{30}$$

Let us write $P(x)\overline{Y_{k,m}(x)}$ in the Gauß decomposition, see Theorem 5.5 in [4], in the form

$$P(x)\overline{Y_{k,m}(x)}=\sum_{j=0}^{N_{k,m}}h_{j,k,m}(x)|x|^{2j}, \tag{31}$$

where $h_{j,k,m}$ are harmonic polynomials and $N_{k,m}$ is the polyharmonic degree of $P(x)Y_{k,m}(x)$. Then (30) and (31) yield

$$\begin{aligned}f_{k,m}(\zeta)&=\frac{1}{\zeta^k}\int_{\mathbb{S}^{n-1}}P(\zeta\theta)\zeta^k\overline{Y_{k,m}(\theta)}\widehat{\mu}(\zeta,\theta)d\theta\\&=\frac{1}{\zeta^k}\sum_{j=0}^{N_{k,m}}\zeta^{2j}\int_{\mathbb{S}^{n-1}}h_{j,k,m}(\zeta\theta)\widehat{\mu}(\zeta,\theta)d\theta\\&=\frac{1}{\zeta^k}\sum_{j=0}^{N_{k,m}}\zeta^{2j}\int_{\mathbb{R}^n}\int_{\mathbb{S}^{n-1}}h_{j,k,m}(\zeta\theta)\frac{1}{\omega_n}\frac{\zeta^{n-1}}{r(\zeta\theta-x)^n}d\theta d\mu(x).\end{aligned}$$

Since $h_{j,k,m}$ is a harmonic polynomial the Poisson formula shows that for real $\zeta>R$ holds

$$h_{j,k,m}(x)=\frac{1}{\omega_n}\int_{\mathbb{S}^{n-1}}h_{j,k,m}(\zeta\theta)\frac{\zeta^{n-2}\left(\zeta^2-|x|^2\right)}{r(\zeta\theta-x)^n}d\theta.$$

Since the integrand is holomorphic in $\zeta$ this holds for all complex values $\zeta$ with $|\zeta|>R$ as well. Thus

$$f_{k,m}(\zeta)=\frac{1}{\zeta^k}\sum_{j=0}^{N_{k,m}}\zeta^{2j}\int_{\mathbb{R}^n}\frac{\zeta}{\zeta^2-|x|^2}h_{j,k,m}(x)d\mu(x) \tag{32}$$

are the Laplace Fourier coefficients of $\theta\mapsto P(\zeta\theta)\widehat{\mu}(\zeta,\theta)$.

Replace now $P(x)\overline{Y_{k,m}(x)}$ in (29) by the right hand side of (31) and take the difference of the Laplace-Fourier coefficients we computed so far. Then the Laplace-Fourier coefficients of $Q_P(\zeta,\theta)$ are given by

$$e_{k,m}(\zeta)=\frac{1}{\zeta^{k-1}}\sum_{j=0}^{N_{k,m}}\int_{\mathbb{R}^n}\frac{1}{\zeta^2-|x|^2}h_{j,k,m}(x)\left(\zeta^{2j}-|x|^{2j}\right)d\mu(x).$$

Note that for $j=0$ the summand ist just zero. For $j\geq 1$ we have

$$\frac{\zeta^{2j}-|x|^{2j}}{\zeta^{2}-|x|^{2}}=|x|^{2(j-1)}+|x|^{2(j-1)}\zeta^{2}+...+\zeta^{2(j-1)}.$$

We conclude that $\zeta\mapsto\zeta^{k-1}e_{k,m}(\zeta)=:P_{k,m}\left(\zeta^{2}\right)$ is a polynomial in $\zeta^{2}$ of degree at most $N_{k,m}-1$. It follows that $e_{k,m}(\zeta)$ can be computed if we know all moments of the form (28) where $t\leq\deg p_{k,m}$ and $h(x)$ is a harmonic polynomial. The proof is complete. ■

From this we have the following interesting consequence

**Corollary 12** *Let $P(x)$ be a polynomial, $\mu$ be a signed measure with support in $\overline{B_R}$ and $Q_P(\zeta,\theta)$ be the corresponding function of the second kind. Then the function $r\theta\mapsto r^{-(n-1)}Q_P(r\theta)$ defined for $r>R$ and $\theta\in\mathbb{S}^{n-1}$, is a polyharmonic function of polyharmonic degree $<N_P$ where $N_P$ is defined in (26).*

**Proof.** By the last theorem the function $\theta\mapsto r^{-(n-1)}Q_P(r\theta)$ has the following Laplace-Fourier expansion

$$f(r\theta):=r^{-(n-1)}Q_P(r\theta)=\sum_{k=0}^{\infty}\sum_{m=1}^{a_k}\frac{1}{r^{n+k-2}}p_{k,m}\left(r^{2}\right)Y_{k,m}(\theta)$$

Let us define the differential operator

$$L_{(k)}:=\frac{d^{2}}{dr^{2}}+\frac{n-1}{r}\frac{d}{dr}-\frac{k(k+n-2)}{r^{2}}. \tag{33}$$

It is known that a function $g(r\theta)$ is a solution of $\Delta^{p}g(x)=0$ if and only if the coefficient functions $g_{k,m}(r)$ of its Laplace-Fourier expansion are solutions of the equation $\left[L_{(k)}\right]^{p}g_{k,m}(r)=0$; an elaboration of these classical results can be found in [21]. Further the polynomials $r^{j}$ with $j=-k-n+2,-k-n+4,...,-k-n+2p$ are solutions of this equation. It follows that

$$f_{k,m}(r)=\frac{1}{r^{n+k-2}}p_{k,m}\left(r^{2}\right)$$

are solutions of the equation $\left[L_{(k)}\right]^{p}g_{k,m}(r)=0$ when $p\geq N_k$. The proof is complete. ■

# 5 Measures with algebraic support

A measure $\mu$ over $\mathbb{R}^{n}$ is *algebraically supported* if the support of the measure is contained in an algebraic set, i.e. if the support of $\mu$ is contained in $P^{-1}(0)$ for some polynomial $P(x)$. This is equivalent to the statement that $\int P^{*}P(x)\,d\mu(x)=0$ where $P^{*}(x):=\overline{P(x)}$ for $x\in\mathbb{R}^{n}$. The Cauchy-Schwarz inequality implies that

$$\left|\int PQd\mu\right|^{2}\leq\int PP^{*}d\mu\cdot\int Q^{*}Qd\mu=0.$$

It follows that $P$ is orthogonal to *all* polynomials $Q$ with respect to $\mu$.

In the one-dimensional case a measure $\mu$ has algebraic support if and only if the support is finite. Further this is equivalent to the property that the Markov transform is a rational function. As we shall see, in the multivariate case all these properties will be different.

**Theorem 13** *Let $\mu$ be a measure with support in $\overline{B_R}$ and let $P(x)$ be a polynomial. Then $\mu$ has support in $P^{-1}\{0\}$ if and only if*

$$P(\zeta\theta)\,\widehat{\mu}(\zeta,\theta) = Q_P(\zeta,\theta) \text{ for all } \theta \in \mathbb{S}^{n-1}, |\zeta| > R, \tag{34}$$

*where $Q_P(\zeta,\theta)$ is the function of the second kind.*

**Proof.** If $\mu$ has support in $P^{-1}\{0\}$ it follows that the rest function $R_P(\zeta,\theta)$ is equal to zero and (34) is evident. For the converse assume that $P(\zeta\theta)\,\widehat{\mu}(\zeta,\theta) = Q_P(\zeta,\theta)$. By Proposition 2 and Theorem 6

$$\begin{aligned}\int P^* P d\mu &= \frac{1}{2\pi i}\int_{\Gamma_{R_1}}\int_{\mathbb{S}^{n-1}} P^*(\zeta\theta)\,P(\zeta\theta)\,\widehat{\mu}(\zeta,\theta)\,d\zeta d\theta \\ &= \frac{1}{2\pi i}\int_{\Gamma_{R_1}}\int_{\mathbb{S}^{n-1}} P^*(\zeta\theta)\,Q_P(\zeta,\theta)\,d\zeta d\theta = 0.\end{aligned}$$

It follows that $\mu$ has support in $P^{-1}\{0\}$. ■

The same proof shows that $\int P^* P d\mu = 0$ if we know that for each fixed $\theta$ the map $\zeta \mapsto P(\zeta\theta)\,\widehat{\mu}(\zeta,\theta)$ is a polynomial in the variable $\zeta$ (since the integral over $\Gamma_{R_1}$ is already zero). Hence we have proved that for a measure $\mu$ with compact support the following implication holds

$$\zeta\widehat{\mu}(\zeta,\theta) \text{ rational} \quad \Rightarrow \quad \operatorname{supp}(\mu) \text{ is contained in an algebraic set,}$$

where rationality of $\widehat{\mu}(\zeta,\theta)$ means that it is a quotient of two polynomial $Q(x)$ and $P(x)$. Not very surprisingly, the converse is not true as the following result shows (where we choose for example $\sigma$ to be equal to the Lebesgue measure on the unit interval):

**Proposition 14** *Let $\sigma$ be a measure $\sigma$ over $\mathbb{R}$ with compact support, $\delta_0$ the Dirac measure over $\mathbb{R}$ at the point $0$ and let $\mu = \sigma \otimes \delta_0$. Then the multivariate Markov transform is given by*

$$\widehat{\sigma \otimes \delta_0}\left(\zeta, e^{it}\right) = \frac{1}{\omega_2}\sum_{l=0}^{\infty}\int x^l d\sigma(x)\,\frac{\sin(l+1)\,t}{\sin t}\frac{1}{\zeta^{l+1}}. \tag{35}$$

*Then $\mu$ has algebraic support but its multivariate Markov transform $\widehat{\sigma \otimes \delta_0}$ is rational if and only if the measure $\sigma$ has finite support.*

**Proof.** Let $\theta = e^{it}$ with $t \in \mathbb{R}$. It is straightforward to verfy that

$$\begin{aligned}\widehat{\sigma \otimes \delta_0}(\zeta, \theta) &= \frac{1}{\omega_2} \int_{\mathbb{R}^2} \frac{\zeta}{r(\zeta\theta - (x,y))^2} d(\sigma \otimes \delta_0) \\ &= \frac{1}{\omega_2} \int_{-\infty}^{\infty} \frac{\zeta}{\zeta^2 - 2\zeta x \cos t + x^2} d\sigma.\end{aligned}$$

Note that

$$\frac{2i\zeta \sin t}{\zeta^2 - 2\zeta x \cos t + x^2} = \frac{1}{\zeta\overline{\theta} - x} - \frac{1}{\zeta\theta - x}.$$

Define for the measure $\sigma$ the one-dimensional Markov transform by $\widetilde{\sigma}(\zeta) = \int \frac{1}{\zeta - x} d\sigma(x)$. Then $2i\omega_2 \sin t \cdot \widehat{\sigma \otimes \delta_0}(\zeta, \theta) = \widetilde{\sigma}\left(\zeta\overline{\theta}\right) - \widetilde{\sigma}(\zeta\theta)$ and the asymptotic expansion of $\widetilde{\sigma}$ leads to (35).

Assume now that $\widehat{\sigma \otimes \delta_0}(\zeta, \theta)$ is rational. Then for $t = \pi/2$ the function $\zeta \mapsto \widehat{\sigma \otimes \delta_0}(\zeta, \theta)$ is rational, i.e. that $f(\zeta) := \sum_{k=0}^{\infty} \int x^{2k} d\mu(x) \frac{1}{\zeta^{2k+1}}$ is a rational function. From the univariate results it follows that $\mu$ must have finite support. ■

If $\mu$ is a measure with finite support and the dimension $n$ is even then it is easy to see that $\zeta\widehat{\mu}(\zeta, \theta)$ is a rational function. The following example shows that the converse is not true:

**Example 15** *Let $\mu$ be the Lebesgue measure on the unit circle $\mathbb{S}^1$. Since the measure is rotation-invariant it follows that $\widehat{\mu}(\zeta, \theta) = \frac{\zeta}{\zeta^2 - 1}$. Hence the multivariate Markov transform $\zeta\widehat{\mu}(\zeta, \theta)$ is a rational function but $\mu$ is not discrete.*

# 6 Proof of Theorem 1

**Proof.** In Theorem 11 we have seen that $Q_{\mu,P}$ and $Q_{\nu,P}$ only depends on the moments $c_{t,k,m}$ where $t < N_P$. It follows that $Q_{\mu,P} = Q_{\nu,P}$. By Theorem 13 $P(\zeta\theta)\widehat{\mu}(\zeta, \theta) = Q_{\mu,P}(\zeta, \theta)$ and $P(\zeta\theta)\widehat{\nu}(\zeta, \theta) = Q_{\nu,P}(\zeta, \theta)$ for all large $\zeta$ and for all $\theta \in \mathbb{S}^{n-1}$, therefore $P(\zeta\theta)\widehat{\mu}(\zeta, \theta) = P(\zeta\theta)\widehat{\nu}(\zeta, \theta)$. We want to conlcude that $\widehat{\mu}(\zeta, \theta) = \widehat{\nu}(\zeta, \theta)$; in that case Theorem 3 yields $\mu = \nu$. If $P(\zeta\theta)$ has no zeros for large $\zeta$ it is clear that $\widehat{\mu}(\zeta, \theta) = \widehat{\nu}(\zeta, \theta)$. In the general case, it suffices to show that $A := \left\{(\zeta, \theta) \in \mathbb{C} \times \mathbb{S}^{n-1} : P(\zeta\theta) = 0\right\}$ is nowhere dense since then a continuity argument leads to $\widehat{\mu}(\zeta, \theta) = \widehat{\nu}(\zeta, \theta)$. This fact will be proven in the next Proposition. ■

Just for completeness sake we include the following

**Proposition 16** *The set $A := \left\{(\zeta, \theta) \in \mathbb{C} \times \mathbb{S}^{n-1} : P(\zeta\theta) = 0\right\}$ is closed and has no interior point, i.e. $A$ is nowhere dense in $\mathbb{C} \times \mathbb{S}^{n-1}$.*

**Proof.** Clearly $A$ is closed. Suppose that there $\theta_0 \in \mathbb{S}^{n-1}$ and $\zeta_0$ such that $P(\zeta\theta) = 0$ for all $\zeta$ in a neighborhood $U$ of $\zeta_0$ and for all $\theta$ in a neighborhood $V$ of $\theta_0$. For fixed $\theta \in V$ it follows that $\zeta \to P(\zeta\theta)$ must be the zero polynomial since for all $\zeta \in U$ (hence uncountably many $\zeta$) we have $P(\zeta\theta) = 0$. It follows

that $P(\zeta\theta) = 0$ for all $\zeta \in \mathbb{C}$ and for all $\theta \in V$. Hence $P(x) = 0$ for all $x$ in an open set $W$ of $\mathbb{R}^n$ and we conclude that $P = 0$. ■

**Corollary 17** *Let $P(x)$ be a polynomial and $N_P$ be given by (26). Then the space*

$$U_{N_P} := \left\{ Q \in \mathcal{P}_n : \ \Delta^{N_P} Q = 0 \right\}$$

*is dense in the space $C(K_P(R), \mathbb{C})$ of all continuous complex-valued functions on $K_P(R)$ endowed with the supremum norm.*

**Proof.** Since $U_{N_P}$ is closed under complex conjugation we may reduce the problem to the case of real-valued continuous functions. Suppose that $U_{N_P}$ is not dense in $C(K_P(R), \mathbb{R})$. By the Hahn-Banach theorem there exists a continuous non-trivial real-valued functional $L$ which vanishes on $U_{N_P}$. By Riesz's Theorem there exists a signed measures $\sigma$ representing the functional $L$ with support in $K_P$. By Theorem 1 we conclude that $\sigma = 0$, a contradiction. ■

# 7 Appendix: The Polyharmonic degree

We want to list some of the properties of the polyharmonic degree map. Note that the inequality $d(P+Q) \leq \max\{d(P), d(Q)\}$ is trivial. In [3] the important equality

$$d\left(Q \cdot |x|^2\right) = d(Q) + d\left(|x|^2\right) = d(Q) + 1. \tag{36}$$

is proved for any polyharmonic function defined on a domain containing zero. The following inequality is implicitly contained in [3, Theorem 1.2, p. 31]. For completeness we give the short proof.

**Proposition 18** *Let $f, g$ be harmonic polynomials. Then $d(ff^*) = \deg f$ and $d(fg) \leq \min\{\deg f, \deg g\}$*

**Proof.** Let $\nabla f$ be the gradient of $f$. Then $\Delta(fg) = (\Delta f)g + 2 < \nabla f, \nabla g > + f\Delta g$. If $h$ and $g$ are harmonic it is easy to show by induction that

$$\Delta^p(fg) = 2^p \sum_{i_1,\dots,i_p=1}^{n} \left(\frac{\partial}{\partial x_{i_1}} \cdots \frac{\partial}{\partial x_{i_p}} f\right)\left(\frac{\partial}{\partial x_{i_1}} \cdots \frac{\partial}{\partial x_{i_p}} g\right).$$

Suppose that $s := \deg f \leq \deg g$. Then $\frac{\partial^\beta}{\partial x^\beta} f = 0$ for all $\beta \in \mathbb{N}_0^n$ with $|\beta| = s+1$. It follows from the above formula that $\Delta^{s+1}(fg) = 0$. Hence $d(fg) = s$. For the first statement note that by the above $d(ff^*) \leq \deg f$. Suppose that $\Delta^{p+1}(ff^*) = 0$ for some $p \in \mathbb{N}$. Then $\sum_{i_1,\dots,i_{p+1}=1}^n \left| \frac{\partial}{\partial x_{i_1}} \cdots \frac{\partial}{\partial x_{i_{p+1}}} f \right|^2 = 0$. It follows that $\frac{\partial^\beta}{\partial x^\beta} f = 0$ for all $\beta \in \mathbb{N}_0^n$ with $|\beta| = p+1$. Hence $\deg f \leq p$ and we have proved that $\deg f \leq d(ff^*)$. The proof is complete. ■

Now we can prove the following

**Corollary 19** *Let $Y_k$ be a harmonic homogeneous polynomial of degree $k$ and $P(x)$ be a polynomial with the Gauß decomposition*

$$P(x) = h_0(x) + |x|^2 h_1(x) + ... + |x|^{2N} h_N(x). \tag{37}$$

*Then*

$$d(P \cdot Y_k) \leq \max_{r=0,\ldots,N} \{r + \deg h_r\} \leq \deg P(x). \tag{38}$$

**Proof.** By (36) $d\left(|x|^{2r} h_r Y_k\right) = r + d(h_r Y_k)$. By Proposition 18 $d(h_r Y_k) \leq \min\{\deg h_r, \deg Y_k\} \leq \deg h_r$. This proves the first inequality. Further we know that $\deg |x|^{2r} h_r = 2r + \deg h_r \leq \deg P$ for $r = 0, ..., N$. Hence the second inequality is established. ■

In the following we want to give an explicit formula for $N_P$.

**Theorem 20** *Let $Y_{k,m}(x)$ be an orthonormal basis of spherical harmonics with $k \in \mathbb{N}_0$ and $m = 1, ..., a_k$. Then $d(Y_{k,m}(x) Y_{k,m_1}(x)) = k$ if and only if $m = m_1$.*

**Proof.** We start with a general remark: Let $Y_k$ and $Y_l$ be harmonic homogeneous polynomials of degree $k$ and $l$ respectively. Clearly $Y_k(x) Y_l(x)$ is a homogeneous polynomial of degree $k+l$. By Proposition 18 it has polyharmonic degree at most $\min\{k,l\}$. By Gauß decomposition there exist harmonic homogeneous polynomials $h_{k+l-2u}$, either $h_{k+l-2u}$ is zero or of exact degree $k+l-2u$ for $u = 0, ..., \min\{k,l\}$, such that

$$Y_k(x) Y_l(x) = \sum_{u=0}^{\min\{k,l\}} |x|^{2u} h_{k+l-2u}(x). \tag{39}$$

Now assume that $Y_k(x) = Y_{k,m}(x)$ and $Y_l(x) = Y_{k,m_1}(x)$. Let us consider the summand $|x|^{2k} h_0(x)$ for $u = k$. Then $h_0$ must have degree 0, hence it is a constant polynomial. Integrate equation (39) with respect to $d\theta$. Since $h_{2k-2u}$ is either 0 or of exact degree $2k - 2u > 0$ for $u = 1, ..., k$ the integral over the sphere of $|x|^{2u} h_{k+l-2u}(x)$ will vanish. Then we obtain

$$\delta_{m,m_1} |x|^{2k} = \int_{\mathbb{S}^{n-1}} h_0 d\theta = h_0 \omega_n.$$

Hence for $m \neq m_1$ we see that the polyharmonic degree is less than $k$, for $m = m_1$ it is exactly $k$. The proof is finished. ■

**Theorem 21** *Let $P(x)$ be a homogeneous polynomial of degree $N$, say of the form*

$$P(x) = \sum_{t,k \in \mathbb{N}_0, 2t+k=N} \sum_{m=1}^{a_k} a_{t,k,m} |x|^{2t} Y_{k,m}(x).$$

*Let $k_0$ be the largest natural number such that $a_{t_0,k_0,m_0} \neq 0$ for some $m_0$ in the above sum. Then*

$$N_P := \sup_{k \in \mathbb{N}_0, m=1,\ldots,a_k} d(P(x) Y_{k,m}(x)) = \frac{1}{2}(N + k_0).$$

**Proof.** Since $d(P+Q) \leq \max\{d(P), d(Q)\}$ we obtain for $k_1 \in N_0$ and $m_1 \in \{1, ..., a_{k_1}\}$ that

$$d\left(P(x) Y_{k_1,m_1}(x)\right) \leq \max d\left(|x|^{2t} Y_{k,m} Y_{k_1,m_1}(x)\right)$$

where the maximum ranges over all indices $t, k, m$ with $a_{t,k,m} \neq 0$. Since $d\left(Y_{k,m} Y_{k_1,m_1}\right) \leq k$ we arrive at (note that $2t+k=N$)

$$d\left(P(x) Y_{k_1,m_1}(x)\right) \leq \max\{t+k\} = \frac{1}{2}\max\{N+k\} \leq \frac{1}{2}(N+k_0).$$

Hence we see that $\frac{1}{2}(N+k_0)$ is a bound for the polyharmonic degree of $P(x) Y_{k_1,m_1}(x)$.

Let us consider $P(x) Y_{k_0,m_0}(x)$ where $k_0$ is as in the theorem. Consider a summand $a_{t,k,m} |x|^{2t} Y_{k,m}$ with $a_{t,k,m} \neq 0$. Then $k \leq k_0$ and Proposition 18 shows that $d\left(Y_{k,m} Y_{k_0,m_0}\right) \leq k$, hence for $k < k_0$ each summand $a_{t,k,m} |x|^{2t} Y_{k,m} Y_{k_0,m_0}$ has polyharmonic degree

$$d\left(a_{t,k,m} |x|^{2t} Y_{k,m} Y_{k_0,m_0}\right) \leq t+k = \frac{1}{2}(N+k) < \frac{1}{2}(N+k_0). \tag{40}$$

Now consider the case $k = k_0$. If $m \neq m_0$ then we apply Theorem 20 and the same argument shows that (40) holds. Finally assume that $k = k_0$ and $m = m_0$. Then Theorem 20 shows that $a_{t_0,k_0,m_0} |x|^{2t_0} Y_{k_0,m_0} Y_{k_0,m_0}$ has exact polyharmonic degree $t_0 + k_0 = \frac{1}{2}(N+k_0)$. Hence we have proven that

$$P(x) Y_{k_0,m_0} = a_{t_0,k_0,m_0} \omega_n |x|^{N+k_0} + R(x)$$

where $R(x)$ has polyharmonic degree $< \frac{1}{2}(N+k_0)$. Thus $P(x) Y_{k_0,m_0}$ has exact polyharmonic degree $\frac{1}{2}(N+k_0)$. ■

Let us finish with the following remark. Let $P(x)$ be an arbitrary polynomial. We can write $P(x) = \sum_{j=0}^{N} P_j(x)$ where $P_j(x)$ are homogeneous polynomials. It is not very difficult to see that

$$d\left(P \cdot Y_{k,m}\right) = \max_{j=0,\ldots,N} d\left(P_j \cdot Y_{k,m}\right),$$

see e.g. the proof of Theorem 1.27 in [4]. Hence $N_P$ is the maximum of $N_{P_j}$ for $j = 0, ..., N$.

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